WHEN IDEAL-BASED ZERO-DIVISOR GRAPHS ARE COMPLEMENTED OR UNIQUELY COMPLEMENTED

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Let R be a commutative ring with nonzero identity and I a proper ideal of R. The *ideal-based zero-divisor graph* of R with respect to the ideal I, denoted by $\Gamma_I(R)$, is the graph on vertices $\{x \in R \setminus I \mid xy \in I \text{ for some } y \in R \setminus I\}$, where distinct vertices x and y are adjacent if and only if $xy \in I$. In this paper,we give a complete classification of when an ideal-based zero-divisor graph of a commutative ring is complemented or uniquely complemented based on the total quotient ring of R/I.

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1. Preliminaries

Let R be a commutative ring with nonzero identity, I a proper ideal of R, and $Z(R)$ the set of zero-divisors of R. Throughout this paper, a *graph* will always be a simple graph, i.e., an undirected graph without multiple edges or loops. In 1988, I. Beck used zero-divisors to produce a graph given a ring R [3]; he was interested in colorings of these graphs. In 1999, D. F. Anderson and P. S. Livingston modified Beck's definition to the following [2,5]; the *zero-divisor graph* of R, denoted by $\Gamma(R)$, is the graph on the vertex set $Z(R)^* = Z(R) \setminus \{0\}$, where two distinct vertices x and y are adjacent if and only if $xy = 0$. In 2001, S. P. Redmond gave the following definition ([7] and [6]) as a generalization of the zero-divisor graph; the graph on vertex set $\{x \in R \setminus I \mid xy \in I \text{ for some } y \in R \setminus I\}$, where distinct vertices x and y are adjacent if and only if $xy \in I$. This is called the *ideal-based zero-divisor graph* of R with respect to the ideal I, denoted by $\Gamma_I(R)$. Note that $\Gamma_I(R)$ and $\Gamma(R/I)$ are non-empty if and only if I is not a prime ideal of R.

Recall that a ring R is von Neumann regular if for every $x \in R$, there exists a $y \in R$ such that $x = xyx$. In [1], the authors find a connection between a ring being von Neumann regular and a graph property called complemented. They define $a \sim b$ if a and b are not adjacent, yet they are adjacent to exactly the same vertices of G. Given distinct vertices a and b of a graph G, we say that the vertices are *orthogonal*, denoted $a \perp b$, if a and b are adjacent and there is no vertex adjacent

to both a and b. Notice that $a \perp b$ if and only if a and b are adjacent and the edge $a - b$ is not part of triangle (a 3-cycle) in G. A graph G is called *complemented* if given any vertex a of G, there exists a vertex b of G such that $a \perp b$. A graph G is *uniquely complemented* if it is complemented and $a \perp b$ and $a \perp c$ imply that $a \sim c$. The preceding relations and definitions are from [1] and [4]. In [1, Theorem 3.5], the authors show that for a reduced ring R, $\Gamma(R)$ is uniquely complemented if and only if $\Gamma(R)$ is complemented, if and only if $T(R)$ is von Neumann regular. In this paper, we extend this result to $\Gamma_I(R)$.

Throughout this paper, R will be a commutative ring with nonzero identity, $Z(R)$ its set of zero-divisors, $nil(R)$ its ideal of nilpotent elements, and total quotient ring $T(R) = R_S$, where $S = R \setminus \{0\}$. Given an ideal I of R, we define $\sqrt{I} = \{r \in R \mid r^k \in I$ for some $k \in \mathbb{N}\}$. A ring R is reduced if $nil(R) = \sqrt{\{0\}} = \{0\}$. Notice that R/I is reduced if and only if $\sqrt{I} = I$. An ideal I is a radical ideal if $\sqrt{I} = I$. Let $\mathbb{Z}$ and $\mathbb{Z}_n$ denote the integers and the integers modulo n, respectively. We will also use the well-known result that $|Z(R)| = 2$ if and only if $R/I \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$. We will denote the set of vertices of a graph G by $V(G)$. In this paper, we will also use that $|V(\Gamma_I(R))| = |I||V(\Gamma(R/I)|$ [6, Corollary 2.7].We say that a graph is complete on n vectrices, denoted by K^n, if it is a graph on n vectiecs in which each vertex is connected to all other vertices.

2. When $\Gamma_I(R)$ is complemented or uniquely complemented

We consider the situation in two cases: either I is a radical ideal of R or I is a non-radical ideal of R.

Proposition 2.1. *Let R be a commutative ring with nonzero identity and I a nonzero, non-radical ideal of R. If $|V(\Gamma(R/I))| \geq 2$, then $\Gamma_I(R)$ is not complemented.*

Proof. Since $I \neq \sqrt{I}$, there exists an $r \in R \setminus I$ such that $r^2 \in I$. Then $r \in V(\Gamma_I(R))$. We claim that r has no complement in $\Gamma_I(R)$. Let s be any vertex of $\Gamma_I(R)$ adjacent to r; so $rs \in I$. Notice that $r \neq s$ as they are distinct adjacent vertices of $\Gamma_I(R)$. Then there are two possibilities: (1) there exists an $i \in I$ such that $s = r + i$ or (2) $s \neq r + i$ for all $i \in I$.

Case (1): Assume there exists an $i \in I$ such that $s = r + i$. Then $r + I = s + I$ in R/I. Since $|V(\Gamma(R/I))| \geq 2$ and $\Gamma(R/I)$ is connected, there exists a vertex $t + I$ adjacent to $r + I = s + I$ in $\Gamma(R/I)$. Notice that $t, r, s = r + i$ are all distinct vertices of $\Gamma_I(R)$ that are mutually adjacent. Thus the edge $r - s$ is part of a triangle in $\Gamma_I(R)$; so s is not a complement of r in $\Gamma_I(R)$.

Case (2): Assume $s \neq r + i$ for all $i \in I$. Since I is non-zero, choose $0 \neq i \in I$. Then the vertices $s, r, r + i$ are distinct mutually adjacent vertices of $\Gamma_I(R)$. Thus the edge $r - s$ is part of a triangle in $\Gamma_I(R)$; so, as before, s is not a complement of r in $\Gamma_I(R)$.

Thus no vertex adjacent to r is a complement of r; so $\Gamma_I(R)$ is not complemented. □

Lemma 2.1. *Let R be a commutative ring with nonzero identity and I an ideal of R. If $\Gamma(R/I) \cong K^1$, then $\Gamma_I(R) \cong K^{|I|}$.*

Proof. $|V(\Gamma(R/I))| = 1$ if and only $|Z(R/I)| = 2$, if and only if $R/I \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$. Thus $V(\Gamma(R/I)) = \{a + I\}$, where $a^2 \in I$. Then $V(\Gamma_I(R)) = \{a + i\}_{i \in I}$. Notice that $(a + i)(a + j) \in I$ for all $i, j \in I$. Moreover $|V(\Gamma_I(R))| = |I||V(\Gamma(R/I))| = |I| \cdot 1 = |I|$. Thus $\Gamma_I(R) \cong K^{|I|}$. $\square$

Theorem 2.1. *Let R be a commutative ring with nonzero identity and I a non-radical ideal of R. Then $\Gamma_I(R)$ is complemented if and only $\Gamma_I(R) \cong K^2$.*

Proof. The "$\Leftarrow$" implication is clear.

Conversely assume that $\Gamma_I(R)$ is complemented. Then $|V(\Gamma(R/I))| \leq 1$ by Proposition 2.1. Since I is not prime (as it is non-radical), it follows that $|V(\Gamma(R/I))| = 1$. Thus $\Gamma_I(R) \cong K^{|I|}$ by Lemma 2.1. Since the only complemented complete graph is K^2, it follows that $|I| = 2$ and $\Gamma_I(R) \cong K^2$. $\square$

Notice that if $|V(\Gamma(R/I))| = 1$, then $R/I \cong \mathbb{Z}_4$ or $\mathbb{Z}_2[X]/(X^2)$; so $\sqrt{I} \neq I$. Moreover, in this case, $\Gamma_I(R)$ is complemented if and only if $|I| = 2$ by the preceding theorem. Thus it remains to investigate the case when $|V(\Gamma(R/I))| \geq 2$.

Theorem 2.2. *Let R be a commutative ring with nonzero identity and I a nonzero, non-prime ideal of R. Then $\Gamma_I(R)$ is complemented and $|V(\Gamma(R/I))| \geq 2$ if and only if $\Gamma(R/I)$ is complemented and $\sqrt{I} = I$.*

Proof. "$\Rightarrow$" Assume that $\Gamma_I(R)$ is complemented and $|V(\Gamma(R/I))| \geq 2$. Then $I = \sqrt{I}$ by Proposition 2.1. So it remains to show that $\Gamma(R/I)$ is complemented. Let $r + I$ be vertex of $\Gamma(R/I)$. Then r is a vertex of $\Gamma_I(R)$. By assumption, $\Gamma_I(R)$ is complemented; so there exists a vertex s of $\Gamma_I(R)$ such that $r \perp s$. We first show that $r+I \neq s+I$. Assume to the contrary; then $r-s = i \in I$. Thus $r(r-s) = ri \in I$. Since $r \perp s$, then $rs \in I$. Hence $r^2 = ri + rs \in I$, and thus $r \in I$ since $\sqrt{I} = I$. This is a contradiction since $r + I \neq I$. Thus $r + I \neq s + I$. Since $r \perp s$ in $\Gamma_I(R)$ and $r + I \neq s + I$, it follows that $r + I$ is adjacent to $s + I$ in $\Gamma(R/I)$. It now remains only to show there is no other vertex in $\Gamma(R/I)$ adjacent to both of these. Assume to the contrary; then there exists a vertex $t + I$ adjacent to both $r + I$ and $s + I$ (hence $t + I, r + I$, and $s + I$ are distinct elements of R/I). Then notice that r, t, s are distinct, mutually adjacent vertices of $\Gamma_I(R)$. But this is a contradiction as $r \perp s$ in $\Gamma_I(R)$. Therefore $r + I \perp s + I$. Since $r + I \in V(\Gamma(R/I))$ was chosen arbitrarily, it follows that $\Gamma(R/I)$ is complemented.

"$\Leftarrow$" Assume that $\Gamma(R/I)$ is complemented and $\sqrt{I} = I$. Since $\Gamma(R/I)$ is complemented and nonempty, it follows that $|V(\Gamma(R/I)| \geq 2$. Let $r \in V(\Gamma_I(R))$; then $r + I \in V(\Gamma(R/I))$. Since $\Gamma(R/I)$ is complemented, there exists a vertex $s + I$ in $\Gamma(R/I)$ such that $r + I \perp s + I$. Since these are vertices in $\Gamma(R/I)$, it follows that neither is zero in R/I; hence $r, s \notin I$ and $rs \in I$. Thus r and s are adjacent

vertices in $\Gamma_I(R)$. We claim that $r \perp s$ in $\Gamma_I(R)$. Assume to the contrary; then there exists a $t \in R \setminus I$ such that r, s, and t are distinct and mutually adjacent in $\Gamma_I(R)$. Using that $\sqrt{I} = I$, a similar argument to that in the forward implication shows that $r + I, s + I$, and $t + I$ are distinct vertices of $\Gamma(R/I)$. It then follows that $r + I, s + I$, and $t + I$ are distinct, mutually adjacent vertices of $\Gamma(R/I)$; but this is a contradiction as $r + I \perp s + I$. Therefore $r \perp s$ in $\Gamma_I(R)$. Since $r \in \Gamma_I(R)$ was chosen arbitrarily, it follows that $\Gamma_I(R)$ is complemented. $\square$

Combining the previous two theorems yields the following result.

Corollary 2.1.
Let R be a commutative ring with nonzero identity and I a proper nonzero non-prime ideal of R. Then $\Gamma_I(R)$ is complemented if and only if exactly one of the following statements holds.

(1) $R/I \cong \mathbb{Z}_4$ or $R/I \cong \mathbb{Z}_2[X]/(X^2)$, and $|I| = 2$.
(2) $\Gamma(R/I)$ is complemented and I is a radical ideal of R.

Using the fact that R/I is reduced if and only if $\sqrt{I} = I$, we can extend the previous theorem to the following corollary using [1, Theorem 3.5].Recall that if I is a prime ideal, then all of the graphs in question are empty. We will consider the empty graph to be vacuously uniquely complemented.

Corollary 2.2. *Let R be a commutative ring with nonzero identity and I a radical ideal of R. Then the following statements are equivalent.*

(1) $\Gamma_I(R)$ is complemented.
(2) $\Gamma(R/I)$ is complemented.
(3) $\Gamma(R/I)$ is uniquely complemented.
(4) $T(R/I)$ is von Neumann regular.

We proceed to consider when $\Gamma_I(R)$ is uniquely complemented. Based on the preceding results, we are led to conjecture that when I is a radical ideal, then $\Gamma_I(R)$ is uniquely complemented if and only $\Gamma_I(R)$ is complemented. The following two lemmas are similar to those found in [7, pp. 55-56].

Lemma 2.2. *Let R be a commutative ring with nonzero identity and I a radical ideal of R. Then $x \perp y$ in $\Gamma_I(R)$ if and only if $x + I \perp y + I$ in $\Gamma(R/I)$.*

Proof. Notice the lemma is vacuously true when $I = \{0\}$. Assume $I \neq \{0\}$. "$\Rightarrow$" First notice that $\sqrt{I} = I$ and $xy \in I$ implies that $x+I \neq y+I$. Otherwise, $y = x+i$ for some $i \in I$. Then $x^2 = x(x + i) - xi = xy - xi \in I$. But $x \in V(\Gamma_I(R))$ implies that $x \notin I$. Hence $x \in \sqrt{I}$ and $x \notin I$, but this is a contradiction as $\sqrt{I} = I$.

Also, $(x + I)(y + I) = 0 + I$, so that $x + I$ and $y + I$ are adjacent vertices of $\Gamma(R/I)$. Assume to the contrary, that there exists $z + I \in V(\Gamma(R/I))$ such that $x+I-y+I-z+I-x+I$ is a triangle in $\Gamma(R/I)$. Then $x-y-z-x$ is a triangle

in $\Gamma_I(R)$, which is a contradiction as $x \perp y$ in $\Gamma_I(R)$. Therefore, $x+I \perp y+I$ in $\Gamma(R/I)$ as desired.

"$\Leftarrow$" Assume that $x+I \perp y+I$ in $\Gamma(R/I)$. Then $xy \in I$; whence x and y are adjacent in $\Gamma_I(R)$. Assume that $x \not\perp y$. Then there exists a vertex c adjacent to both x and y in $\Gamma_I(R)$. We claim that then $c+I$ is distinct from $x+I$ and $y+I$ and each of these three vecrtices are adjacent to each other. To see that $c+I$ is distinct from $x+I$ and $y+I$, assume to the contrary. Without loss of generality, assume $c+I = x+I$. Then $c = x+i$ for some $i \in I$. Then $cx \in I$ implies that $x^2 \in I$, which is a contradiction as $\sqrt{I} = I$ and $x+I$ is nonzero. Since $x+I$, $y+I$, and $c+I$ are distinct and xy, yc, and $xc \in I$, it follows that $x+I$, $y+I$, and $c+I$ is a three-cycle in $\Gamma(R/I)$. But this is a contradiction as $x+I \perp y+I$ in $\Gamma(R/I)$. □

Lemma 2.3. *Let R be a commutative ring with nonzero identity and I a radical ideal of R. If $\Gamma(R/I)$ is uniquely complemented, $x \perp y$ and $x \perp z$ in $\Gamma_I(R)$, and $\alpha \in R \setminus I$, then*

$$\alpha y \in I \text{ if and only if } \alpha z \in I.$$

Proof. The statement is symmetric in terms of y and z; so it suffices to show that $\alpha y \in I \Rightarrow \alpha z \in I$.

By Lemma 2.2, $x+I \perp y+I$ and $x+I \perp z+I$ in $\Gamma(R/I)$. Since $\Gamma(R/I)$ is uniquely complemented, it follows that $\mathrm{ann}_{R/I}(y+I) = \mathrm{ann}_{R/I}(z+I)$ (here we also using the fact $\mathrm{ann}_{R/I}(y+I) \setminus \{y+I\} = \mathrm{ann}_{R/I}(y+I)$ and $\mathrm{ann}_{R/I}(x+I) \setminus \{x+I\} = \mathrm{ann}_{R/I}(x+I)$ since $\sqrt{I} = I$).

Assume $\alpha y \in I$. Then $\alpha + I \in \mathrm{ann}_{R/I}(y+I) = \mathrm{ann}_{R/I}(z+I)$. Hence $(\alpha+I)(z+I) = 0+I$, and therefore $\alpha z \in I$ as desired. □

Theorem 2.3. *Let R be a commutative ring with nonzero identity and I a radical ideal of R. Then $\Gamma_I(R)$ is complemented if and only if $\Gamma_I(R)$ is uniquely complemented.*

Proof. If $I = (0)$, then the result follows from [1, Theorem 3.5]. If $\Gamma_I(R)$ is the empty graph, the statement holds vacuously. Assume that $I \neq (0)$ and that $\Gamma_I(R)$ is not the empty graph (i.e., I is not a prime ideal of R).

The reverse implication is by definition.

Assume $\Gamma_I(R)$ is complemented. Then $\Gamma_I(R)$ has at least two elements, and thus $V(\Gamma(R/I))$ must be nonempty. Since I is a radical ideal, it follows that $|V(\Gamma(R/I))| \neq 1$ (since there are only two rings up to isomorphism with exactly 2 zero-divisors, and they are both non-reduced rings). Thus $|V(\Gamma(R/I))| \geq 2$, and hence $\Gamma(R/I)$ is complemented by Theorem 2.2. Moreover, $\Gamma(R/I)$ is uniquely complemented by Corollary 2.2. The desired result then follows from Lemma 2.3. □

Theorem 2.4. *Let R be a commutative ring with nonzero identity and I a proper radical ideal of R. Then the following statements are equivalent.*

(1) $\Gamma_I(R)$ *is complemented.*
(2) $\Gamma_I(R)$ *is uniquely complemented.*
(3) $\Gamma(R/I)$ *is complemented.*
(4) $\Gamma(R/I)$ *is uniquely complemented.*
(5) $T(R/I)$ *is von Neumann regular.*

Moreover, regardless if I *is a radical or non-radical ideal,* $\Gamma_I(R)$ *is complemented if and only if* $\Gamma_I(R)$ *is uniquely complemented.*

Proof. If I is a prime ideal ideal of R, then all of the graphs in question are empty and R/I is an integral domain. Thus all of the conditions hold.

If $I = (0)$ and radical, then the theorem holds by [1, Theorem 3.5]; in this case, the conditions (1) and (3) are equivalent as are conditions (2) and (4).

Assume that I is a nonzero, proper, non-prime, radical ideal of R. The equivalences follow from Corollary 2.2 and Theorem 2.3.

For the "moreover statement," if I is not a radical ideal, then $\Gamma_I(R)$ is complemented if and only if $\Gamma_I(R) \cong K^2$ by Theorem 2.1. However, K^2 is uniquely complemented. Thus, regardless of whether or not I is a radical ideal of R, we have $\Gamma_I(R)$ is uniquely complemented if and only if $\Gamma_I(R)$ is complemented. $\square$

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